# Conservation of space-charge-dominated beam emittance in a strong nonlinear focusing field

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Beam dynamics in a nonlinear uniform focusing channel is studied from the viewpoint of keeping emittance of a high current beam. Conservation of beam emittance is treated as a problem of proper matching of the beam with the uniform focusing channel. To obtain matching conditions for a beam with an arbitrary distribution function, it is necessary to accept that the potential of the external focusing field contains higher-order terms than quadratic. The solution for the external potential is obtained from the stationary Vlasov's equation for the beam distribution function and Poisson's equation for the electrostatic beam potential. An analytical approach is illustrated by results of a particle-in-cell simulation. [S1063-651X(96)09105-2]

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## I. INTRODUCTION

The nonlinear space-charge field of a beam is a serious concern for beam emittance growth in particle acceleration facilities. This effect is most pronounced when particles are slow and space-charge forces are significant. The problem of beam emittance growth due to a nonstationary beam profile in a focusing channel with a linear focusing field has been treated in many papers (see Refs. [1–12], and cited references there). The general property of space-charge-dominated beam behavior is that a beam with an initial nonlinear profile tends to become more uniform and this process is associated with strong emittance growth and the appearance of beam halo. In Fig. 1 an example of beam dynamics with an initial Gaussian profile in a uniform focusing channel is presented. After a few transverse oscillations, the mismatching of the initial beam profile results in the appearance of a uniform beam core accompanied by a halo formation.

The beam emittance is conserved if the beam is matched with the channel. The problem of matching of the nonlinear density profiled beam with a linear uniform focusing channel was studied in detail in Refs. [12–15]. The analytical approach is based on the fact that the Hamiltonian of the matched beam is a constant of motion, and therefore the unknown distribution function can be expressed as a function of the Hamiltonian:

$$f(x,p_x,y,p_y)=f(H), \tag{1}$$

$$H=\frac{p_x^2+p_y^2}{2m}+\frac{k^2}{2}(x^2+y^2)+qU_b(x,y). \tag{2}$$

In Eq. (2) the parameter $k^2$ describes the focusing of particles in solenoids or a smoothed external focusing in an alternating-gradient structure and $U_b(x,y)$ is the space-charge potential of the beam. Combination of Eqs. (1) and (2) with Poisson's equation gives the integral equation for the self-consistent space-charge potential of a beam in a focusing channel:

$$\Delta U_b=-\frac{q}{\varepsilon_0}\int f\left(\frac{p_x^2+p_y^2}{2m}+\frac{k^2}{2}(x^2+y^2)+qU_b\right)dp_x\,dp_y, \tag{3}$$

where $q$ is a charge of particles and $\varepsilon_0=8.85\times10^{-12}$ F/m is the permittivity of vacuum. After finding the space-charge potential of the beam $U_b$, the self-consistent distribution function can be found using Eq. (1). A general property of the solution is that with increasing beam current, the profile of the matched beam has to be more and more flat while the phase space projection (beam emittance) has to be more and more close to a rectangle.

Laboratory beams are usually far from the above solution and suffer serious emittance growth (see Fig. 1). It is interesting to verify whether it is possible to match a realistic beam with a focusing channel. Instead of finding a self-consistent distribution in the linear focusing channel one can try to adjust the external potential in such a way that the given beam distribution will be preserved. As shown in Ref. [16], matching of a realistic beam with a uniform focusing channel can be achieved if the focusing field is not linear anymore. In this paper we present the required focusing potential as a series in power of beam current.

## II. MATCHING OF THE BEAM WITH ARBITRARY DISTRIBUTION FUNCTION

To find the matching conditions for a beam with an arbitrary distribution function, let us assume as in Eqs. (1) and (2) that the beam is matched with the channel. Hence, the Hamiltonian is a constant of motion but no assumptions about linearity of focusing forces are adopted:

$$H=\frac{p_x^2+p_y^2}{2m}+qU(x,y)=\text{const.} \tag{4}$$

The total potential of the structure is a combination of the external focusing potential $U_{\text{ext}}$ and the space-charge poten-

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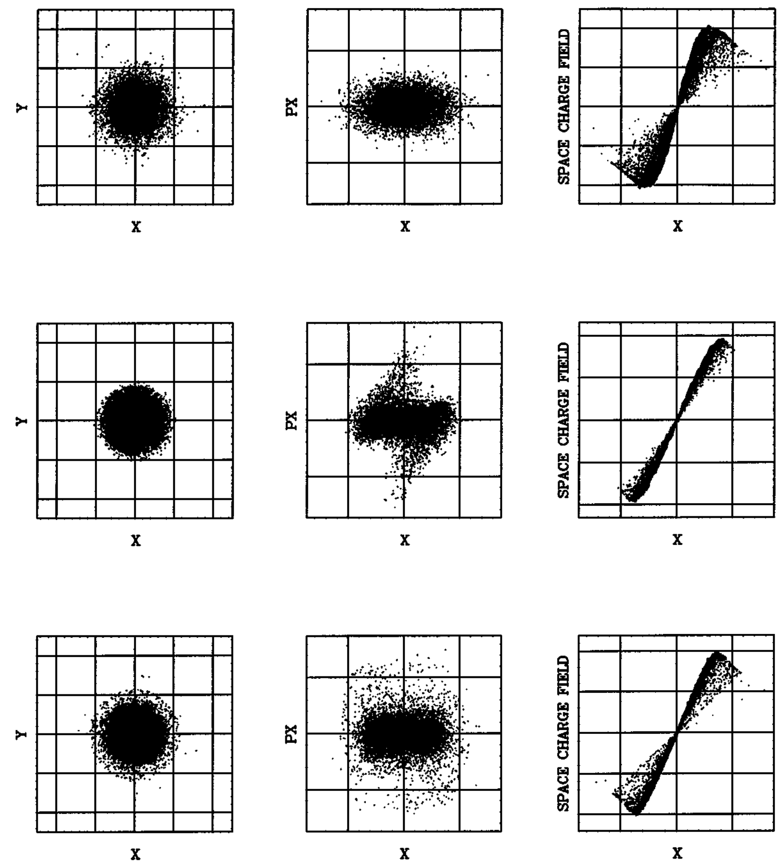


FIG. 1. Beam profile (left column), phase space projections (middle column), and space-charge forces (right column) of a nonstationary beam in the linear focusing channel. Discrepancy between the space-charge forces for nearby particles comes from projection of radial forces on the $x$ axis, $E_x=E(r)x/r$, where the ratio $E(r)/r$ is different for particles with fixed position $x$.

tial $U_b$ of the beam, $U=U_{\text{ext}}+U_b$. The time-independent distribution function of a matched beam obeys Vlasov's equation:

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\,v_x+\frac{\partial f}{\partial y}\,v_y-q\left(\frac{\partial f}{\partial p_x}\,\frac{\partial U}{\partial x}+\frac{\partial f}{\partial p_y}\,\frac{\partial U}{\partial y}\right)=0, \tag{5}$$

where the partial derivative of the distribution function over time is omitted due to initial matched conditions. The distribution function of the beam is supposed to be given from the source of particles of the beam. Therefore, the self-potential of the beam $U_b$ is also a known function derived from Poisson's equation:

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial U_b}{\partial r}\right)=-\frac{\rho(r)}{\varepsilon_0}, \tag{6}$$

where $\rho(r)$ is the space-charge density of the beam. Combining solutions of Vlasov's equation for the total potential of the structure, $U$, and space-charge potential of the beam, $U_b$, obtained from Poisson's equation, the external potential of the focusing structure can be found:

$$U_{\text{ext}}=U-U_b. \tag{7}$$

The solution of this problem has to be found for every specific particle distribution.

## III. GAUSSIAN BEAM MATCHED WITH THE CHANNEL

Let us consider a $z$-uniform beam with a Gaussian distribution function in four-dimensional phase space, which is close to the experimentally observed beam distribution:

$$f=f_0\exp\left(-2\,\frac{x^2+y^2}{R^2}-2\,\frac{p_x^2+p_y^2}{p_0^2}\right). \tag{8}$$

This distribution makes an elliptical phase space projection at every phase plane with normalized root-mean-square (rms) beam emittance:

$$\varepsilon=\frac{4}{mc}\,\sqrt{\langle x^2\rangle\langle p_x^2\rangle-\langle xp_x\rangle^2}=R\,\frac{p_0}{mc}. \tag{9}$$

Substituting the distribution function (8) into Vlasov's equation yields an expression for the total unknown potential of the structure:

$$\frac{mc^2}{q}\,(xp_x+yp_y)=\frac{R^4}{\varepsilon^2}\left(p_x\,\frac{\partial U}{\partial x}+p_y\,\frac{\partial U}{\partial y}\right). \tag{10}$$

Vlasov's equation can be separated into two independent parts for $x$ and $y$ coordinates, respectively:

$$\frac{\partial U}{\partial x}=\frac{mc^2\varepsilon^2}{qR^4}\,x,\quad \frac{\partial U}{\partial y}=\frac{mc^2\varepsilon^2}{qR^4}\,y. \tag{11}$$

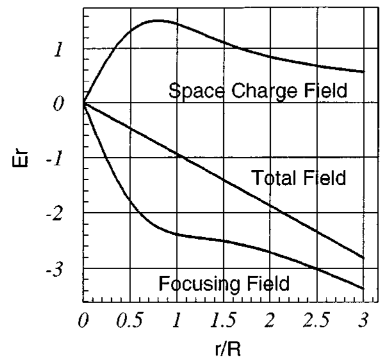


FIG. 2. Total field of the structure $E_{\text{tot}}$ [Eq. (12)], required external focusing field $E_{\text{ext}}$ [Eq. (15)], and space-charge field of the Gaussian beam $E_b$ [Eq. (14)].

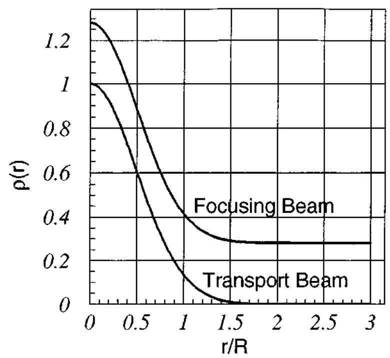


FIG. 3. Charged particle density of the transport beam with Gaussian distribution [Eq. (13)] and of the external focusing beam [Eq. (17)].

Combining solutions of Eq. (11), the total potential of the structure is a quadratic function of coordinates, which creates linear focusing field $E_{\text{tot}}$:

$$U(x,y)=\frac{mc^2}{q}\frac{\varepsilon^2}{R^4}\left(\frac{x^2+y^2}{2}\right),$$

$$E_{\text{tot}}=-\frac{\partial U}{\partial r}=-\frac{mc^2}{q}\frac{\varepsilon^2}{R^4}r. \tag{12}$$

The appearance of quadratic terms in the total potential of the structure is quite clear because phase space projections of the beam have elliptical shape and an ellipse is conserved in a linear field. The space-charge field of the beam $E_b$ is calculated from Poisson's equation using a known space-charge density function of the beam $\rho_b$:

$$\rho_b=\rho_0\exp\left(-2\frac{r^2}{R^2}\right), \tag{13}$$

$$E_b=-\frac{\partial U_b}{\partial r}=\frac{I}{2\pi\varepsilon_0\beta c}\frac{1}{r}\left[1-\exp\left(-2\frac{r^2}{R^2}\right)\right], \tag{14}$$

where $\rho_0=2I/(\pi c\beta R^2)$, $I$ is the beam current, and $\beta$ is the longitudinal velocity of particles. Subtraction of the space-charge field from the total field of the structure gives the expression for the external focusing field of the structure, which is required for conservation of beam emittance:

$$E_{\text{ext}}=-\frac{mc^2}{qR}\left\{\frac{\varepsilon^2 r}{R^3}+2\frac{I}{I_c\beta}\frac{R}{r}\left[1-\exp\left(-2\frac{r^2}{R^2}\right)\right]\right\}, \tag{15}$$

where $I_c=4\pi\varepsilon_0 mc^3/q=A/Z\times 3.13\times 10^7$ A is a characteristic value of the beam current. The relevant potential of the focusing field is given by the expression

$$U_{\text{ext}}(r)=\frac{mc^2}{q}\left[\left(\frac{\varepsilon^2}{2R^4}+\frac{2I}{I_c\beta R^2}\right)r^2+\frac{2I}{I_c\beta}\left(-\frac{r^4}{2R^4}+\frac{2}{9}\frac{r^6}{R^6}+\cdots+\frac{(-1)^{k+1}2^k r^{2k}}{2kk!R^{2k}}\right)\right]. \tag{16}$$

Let us note that the external potential of the structure consists of two parts: quadratic (which produces linear focusing) and higher order terms that describe nonlinear focusing. The linear part depends on the values of beam emittance and beam current while the nonlinear part depends on beam current only. This means that the external field has to compensate the nonlinearity of self-field of the beam and produce required linear focusing of the beam to keep the elliptical beam phase space distribution. Figure 2 illustrates the relationships between space-charge field of the beam, total field, and focusing field of the structure. The external focusing field obtained from the above consideration is a complicated function of radius, which is linear near the axis and becomes nonlinear far from the axis. One of the ways to create the required focusing potential is to introduce inside the transport channel an opposite charged cloud of particles (plasma lens) with the space-charge density:

$$\rho_{\text{ext}}=\rho_0\exp\left(-2\frac{r^2}{R^2}\right)+\frac{I_c\varepsilon^2}{2\pi cR^4}. \tag{17}$$

In Fig. 3 the charged particle density of the transport beam and the external focusing beam are presented.

## IV. ‘‘WATER BAG’’ AND ‘‘PARABOLIC’’ BEAM MATCHED WITH THE CHANNEL

The analogous result can be obtained for a beam with other distributions with elliptical symmetry. Let us consider uniformly populated four-dimensional (4D) hypervolume in four-dimensional phase space, which is called the ‘‘water bag’’ distribution [2]:

$$f=f_0,\quad \frac{2}{3}\left(\frac{x^2+y^2}{R^2}+\frac{p_x^2+p_y^2}{p_0^2}\right)\leqslant 1,$$
$$f=0,\quad \frac{2}{3}\left(\frac{x^2+y^2}{R^2}+\frac{p_x^2+p_y^2}{p_0^2}\right)>1. \tag{18}$$

The coefficient 2/3 in Eq. (18) is chosen from normalization of the distribution and reflects the fact that the maximum beam sizes for such a distribution are $\sqrt{3/2}$ larger than rms

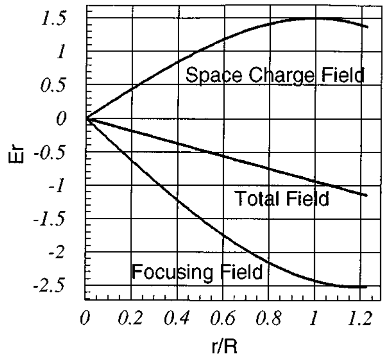


FIG. 4. Total field of the structure $E_{tot}$ [Eq. (12)], required external focusing field $E_{ext}$ [Eq. (21)] and space-charge field $E_b$ [Eq. (20)] of the beam with ''water bag'' distribution.

beam parameters $R$ and $p_0$. This distribution is characterized by a parabolic space-charge density function in real space:

$$\rho(r)=\frac{4I}{3\pi\beta cR^2}\left(1-\frac{2r^2}{3R^2}\right). \tag{19}$$

The solution of Vlasov's equation is the same as for the potential described by Eq. (12). The space-charge field of the beam is a two-term function of radius

$$E_b=\frac{2I}{3\pi\varepsilon_0\beta c}\frac{r}{R^2}\left(1-\frac{r^2}{3R^2}\right). \tag{20}$$

The corresponding external focusing field is given by the expression

$$E_{ext}=-\frac{mc^2r}{qR^2}\left[\frac{\varepsilon^2}{R^2}+\frac{8I}{3I_c\beta R^2}\left(1-\frac{r^2}{3R^2}\right)\right]. \tag{21}$$

In Fig. 4 the space-charge potential, total field, and required focusing field of the structure or the beam with the water bag distribution are presented. As in the case of a Gaussian beam the required focusing field is close to a linear function of radius near the axis and drops nonlinearly far from the axis.

For the ''parabolic'' distribution [2] phase space density of particles monotonically decreases from the center of the beam until the boundary of four-dimensional hypervolume:

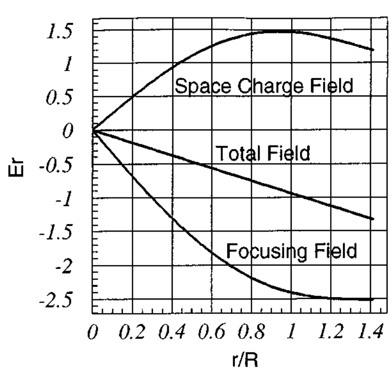


FIG. 5. Total field of the structure $E_{tot}$ [Eq. (12)], required external focusing field $E_{ext}$ [Eq. (25)] and space-charge field $E_b$ [Eq. (24)] of the beam with parabolic distribution.

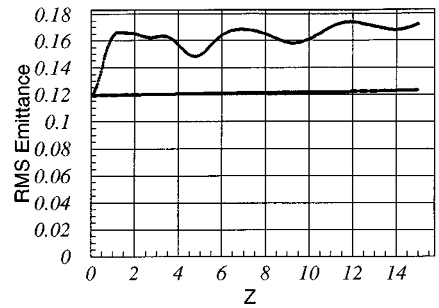


FIG. 6. Emittance growth of the Gaussian beam in the linear focusing channel (upper curve) and emittance conservation in nonlinear focusing channel (lower curve).

$$f=f_0\left(1-\frac{x^2+y^2}{2R^2}-\frac{p_x^2+p_y^2}{2p_0^2}\right). \tag{22}$$

Maximum beam sizes for such a distribution are $\sqrt{2}$ larger than rms beam parameters $R$ and $p_0$, which is reflected in the coefficient 2 in the denominator of the distribution in Eq. (22). The space-charge density function of the beam $\rho_b$ and electrical field of the beam $E_b$ are defined by the following expressions:

$$\rho_b=\frac{3I}{2\pi c\beta R^2}\left(1-\frac{r^2}{2R^2}\right)^2, \tag{23}$$

$$E_b=\frac{3Ir}{4\pi\varepsilon_0\beta cR^2}\left[1-\left(\frac{r}{\sqrt{2}R}\right)^2+\frac{1}{3}\left(\frac{r}{\sqrt{2}R}\right)^4\right]. \tag{24}$$

The relevant focusing field that is required to conserve beam emittance is given by the expression

$$E_{ext}=-\frac{mc^2r}{qR^2}\left[\frac{\varepsilon^2}{R^2}+\frac{3I}{I_c\beta}\left(1-\frac{r^2}{2R^2}+\frac{r^4}{12R^4}\right)\right]. \tag{25}$$

In Fig. 5 the space-charge field, total field, and required focusing field of the structure for the beam with the parabolic distribution are presented.

### V. RESULTS OF PARTICLE-IN-CELL SIMULATIONS

To verify the possibility of conservation of beam emittance in a nonlinear focusing field, a beam dynamics simulation using particle-in-cell code BEAMPATH [17] has been performed. A beam of particles is represented as a collection of large number (usually $1.3\times10^4$) trajectories. Equations of motion are integrated using a time-centered second-order integrator with constant time step $\Delta t$ (''leap-frog'' method) [18]:

$$\vec{p}_{i+1/2}=\vec{p}_{i-1/2}+q\vec{E}_i\Delta t,$$
$$\vec{r}_{i+1}=\vec{r}_i+\vec{v}_{i+1/2}\Delta t. \tag{26}$$

The value of discrete time step $\Delta t$ in simulation is chosen small enough ($10^{-2},...,10^{-3}$ of particle oscillation period) that the results of simulations are insensitive to the changes

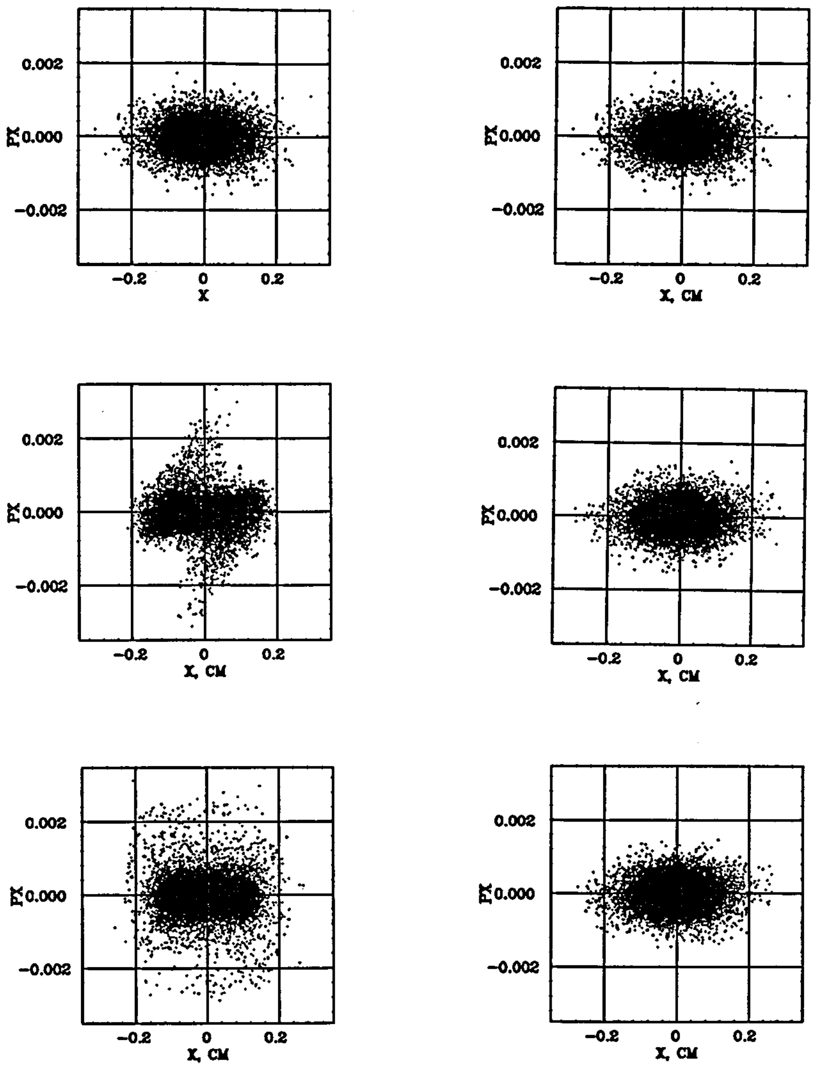


FIG. 7. Mismatching of the Gaussian beam in the linear focusing channel (left column) and matching of the same beam with the nonlinear focusing channel (right column).

of $\Delta t$. The space-charge field of a $z$-uniform beam is found from a two-dimensional Poisson's equation in Cartesian coordinates:

$$\frac{\partial^2 U_b}{\partial x^2}+\frac{\partial^2 U_b}{\partial y^2}=-\frac{\rho(x,y)}{\varepsilon_0}. \tag{27}$$

The Dirichlet boundary condition for potential $U_b$ is imposed on the surface of an infinite rectangular pipe with transverse sizes $a\times a$. The region occupied by an ensemble of particles is divided into uniform rectangular meshes of dimension $NX\times NY=256\times256$. The charge of every particle is distributed among the nearest four nodes inversely proportional to the distance of the particle from each node. To obtain a solution of Poisson's equation, the space-charge density of the beam and the unknown potential functions are represented as Fourier series:

$$\rho_{ij}=\sum_{u=1}^{NX-1}\sum_{v=1}^{NY-1}\overline{\rho_{u,v}}\sin\left(\frac{\pi u i}{NX}\right)\sin\left(\frac{\pi v j}{NY}\right), \tag{28}$$

$$U_{ij}=\sum_{u=1}^{NX-1}\sum_{v=1}^{NY-1}\overline{U_{u,v}}\sin\left(\frac{\pi u i}{NX}\right)\sin\left(\frac{\pi v j}{NY}\right). \tag{29}$$

Calculation of the series is performed using a fast Fourier transformation method. Space-charge and potential expansion coefficients are connected by an algebraic relationship following Poisson's equation:

$$\overline{U_{uv}}=\frac{\overline{\rho_{uv}}}{\varepsilon_0[(\pi u/a)^2+(\pi v/a)^2]}, \tag{30}$$

which gives the solution of the space-charge problem. Electric field components are calculated by numerical differentiation of the potential grid function.

In Figs. 6 and 7 the results of the beam dynamics study with initial Gaussian distribution in linear and nonlinear focusing channels are presented. Parameters of the beam were chosen as follows: $A/Z=1$, $I=2$ A, $\varepsilon=0.12\pi$ cm mrad, $R=0.15$ cm, $\beta=0.0178$. The external focusing potential for the linear focusing channel was taken as

$$U_{\rm ext}(r)=\frac{mc^2}{q}\left(\frac{\varepsilon^2}{2R^4}+\frac{I}{I_c\beta R^2}\right)r^2, \tag{31}$$

which corresponds to the matched conditions for an equivalent Kapchinsky-Vladimirsky (KV) [13] beam with the same rms beam emittance $\varepsilon$ and rms beam size $R$. In the case of nonlinear focusing, the external potential is represented by Eq. (16). Let us note that quadratic terms in potentials (16) and (31) are different.

From results of simulations, it is seen that in both cases the sizes of the beam in real space are close to constant, which is typical for matching of the beam, taking into account rms beam sizes. But in the case of linear focusing, the beam is mismatched in the phase plane, which results in 50% emittance growth accompanied by halo formation. At the same time the beam is completely matched with the nonlinear focusing channel, and this results in conservation of all beam characteristics and does not suffer any serious emittance growth.

## VI. FOCUSING BY A STATIC FIELD

The required external focusing field obtained from the above consideration is a complicated function of radius, which is linear near the axis and drops nonlinearly far from the axis. This specific feature of the focusing field restricts the possible ways to produce the appropriate potential distribution. Axial-symmetric electrostatic and magnetostatic lenses have aberrations that increase the focusing of charged particles with radius as compared with linear focusing [12]. A time-independent field provides a focusing effect that can be described by a linear term as well as higher-order terms. The paraxial equation of radial motion of a particle in the electrostatic lens with the field distribution along the axis $E_z(z)$ is given by

$$\frac{d^2r}{dt^2}=\frac{P_\theta^2}{m^2r^3}-\frac{q}{m}\left[\frac{r}{2}\frac{\partial E_z}{\partial z}-\frac{r^3}{16}\frac{\partial^3 E_z}{\partial z^3}+\cdots\right]. \tag{32}$$

where $P_\theta$ is an azimuth component of canonical momentum of particle. After passing through the lens the slope of the particle trajectory $r$ is changed as follows:

$$\Delta r' = -\frac{r}{f}(1+C_\alpha r^2), \tag{33}$$

where $f$ is a focal length of the lens and $C_\alpha$ is a spherical aberration coefficient. From Eq. (33) it follows that the changing of slope of trajectory is larger for particles with larger radius. Spherical aberrations of axial-symmetric lenses result in hollow beam profile formation and emittance growth [12].

Most of the focusing channels are based on alternating-gradient principle employing alternating focusing-defocusing quadrupole lenses with linear focusing field distribution across the aperture. The higher-order multipole lenses (sextupoles, octupoles, etc.) create essentially nonlinear field due to azimuth variation of potential $U_{\rm ext}=U_0 r^n \cos n\theta$. Focusing and defocusing directions are repeated after azimuth angle shift $\Delta\theta=\pi/n$. The potential of the quadrupole alternating-gradient focusing channel is presented as follows:

$$U(r,\varphi,z)=\frac{G_2(z)}{2}r^2\sin 2\varphi+\frac{G_6(z)}{6}r^6\sin 6\varphi+\cdots\ , \tag{34}$$

where $G_2(z)$ is a quadrupole gradient and $G_6(z)$ is a duodecapole component. To create the required nonlinear compensation of space-charge field in two orthogonal $x$-$y$ directions the sign of the duodecapole component $G_6(z)$ should be opposite to the sign of the quadrupole component $G_2(z)$. Let us consider the one-dimensional problem for a particle oscillating in the field (34):

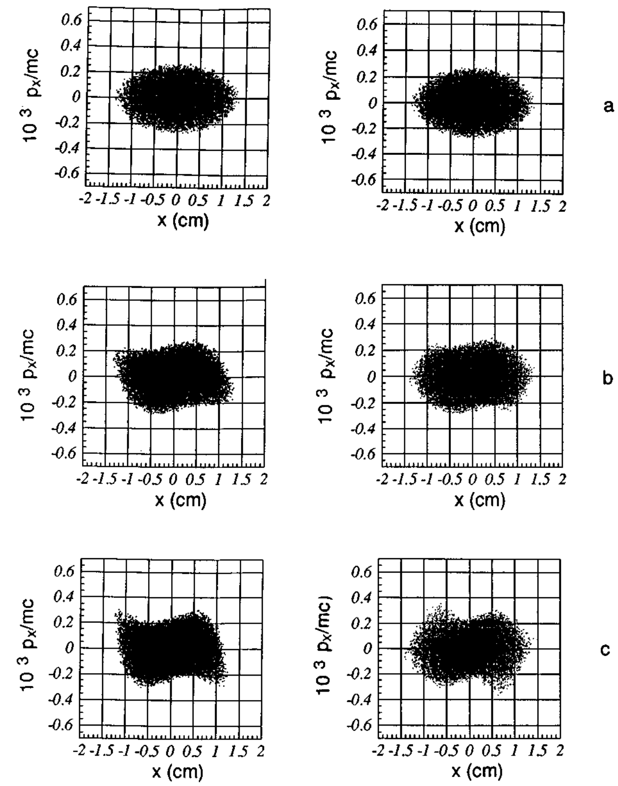


FIG. 8. Phase space projections of the beam in the pure quadrupole FD structure (left column) and in quadrupole structure with duodecapole component (right column): (a) initial beam; (b) after 10 lenses; (c) after 20 lenses.

$$m\frac{d^2x}{dt^2}=q[G_2(z)x+G_6(z)x^5]. \tag{35}$$

We restrict our consideration to a FD (focusing-defocusing) structure, where all lenses have the same length $D$ and the period of the structure is $S=2D$. The solution of the problem can be represented as a combination of the slow varying deviation of the particle from axis $X$ and the fast oscillating variable $\xi$. Employing an averaging method [19] one can obtain the following equation for slow variable $X$, which is essential for the definition of the focusing properties of the channel:

$$\frac{d^2X}{d\tau^2}+\mu_0^2\left[X+6\frac{G_6}{G_2}X^5\right]=0, \tag{36}$$

where $\tau=z/S$ is a dimensionless longitudinal coordinate and $\mu_0$ is a frequency of the smoothed oscillations in the FD structure:

$$\mu_0=\frac{G_2D^2}{\sqrt{2}(mc^2/q)\beta^2}. \tag{37}$$

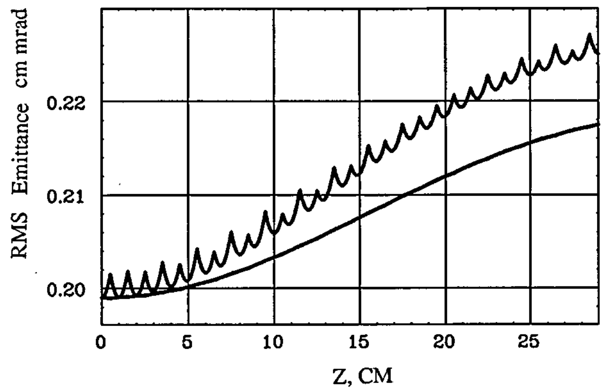


FIG. 9. rms beam emittance growth in the quadrupole structure with the duodecapole component (upper curve) and in the pure quadrupole structure (lower curve).

Let us consider a beam with the parabolic distribution (22). The field required to conserve beam emittance for parabolic particle distributions is given by Eq. (25). It consists of terms proportional to $r$, $r^3$, and $r^5$, while the field of the FD structure consists of terms $x, x^5$. Let us choose the parameters of quadrupole structure from the conditions that fields (25) and (36) are equal to each other near the axis, where only linear focusing terms are essential, and at the boundary of the beam at $r=\sqrt{2}R$. It gives the following expressions for the field gradient and for the duodecapole component:

$$G_2=\sqrt{8}\,\frac{mc^2\beta}{qRD}\left(\frac{\varepsilon^2}{R^2}+\frac{3I}{I_c\beta}\right)^{1/2}, \tag{38}$$

$$G_6=-\frac{1}{3}\,\frac{I}{I_c}\,G_2\,\frac{D^2}{R^6}\,\frac{1}{\mu_0^2\beta^3}\,. \tag{39}$$

We present in Figs. 8 and 9 the results of beam dynamics simulation in the FD structure with $G_2=36$ kV/cm$^2$, $G_6=-1$ kV/cm$^6$; $D=1$ cm for the beam with $A/Z=1$, $W=150$ keV, $I=100$ mA, $\varepsilon=0.2\pi$ cm mrad, $R=1$ cm. As shown, the beam emittance shape is better conserved in the focusing channel with the duodecapole component while the rms emittance growth is smaller in the channel with the pure quadrupole field. It confirms the assumption that the nonlinear focusing field component compensates nonlinear space-charge field but at the same time creates strong $x$-$y$ coupling, which itself is a source of emittance growth. The beam with arbitrary nonlinear distribution cannot be exactly matched with the quadrupole channel, but better matching as far as the whole phase space area occupied by the beam is concerned can be achieved in the channel with the nonlinear focusing component. Strong $x$-$y$ coupling arising from the duodecapole component does not allow one to use this field for beam matching with the high value of phase space density. In numerical experiments the effect of nonlinear space-charge field compensation and, therefore, beam matching was observed for beams with the value of phase space density no more than 0.6 A/cm mrad.

## VII. CONCLUSIONS

Conservation of beam emittance was treated as a problem of proper matching of the beam with a uniform focusing channel. Matched conditions for the beam with elliptical phase space projections but nonlinear space-charge forces in a uniform focusing channel require the focusing field to include nonlinear terms of higher order than quadratic. The solution for the external potential is attained from the stationary Vlasov's equation for the beam distribution function and Poisson's equation for the electrostatic beam potential. The focusing field produces linear focusing near the axis of the structure but has to change nonlinearly away from the axis. Different examples of Gaussian, ''water bag'' and ''parabolic'' distributions in 4D phase space are considered. Results of a particle-in-cell simulation confirm the conservation of beam emittance in a nonlinear external field.


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